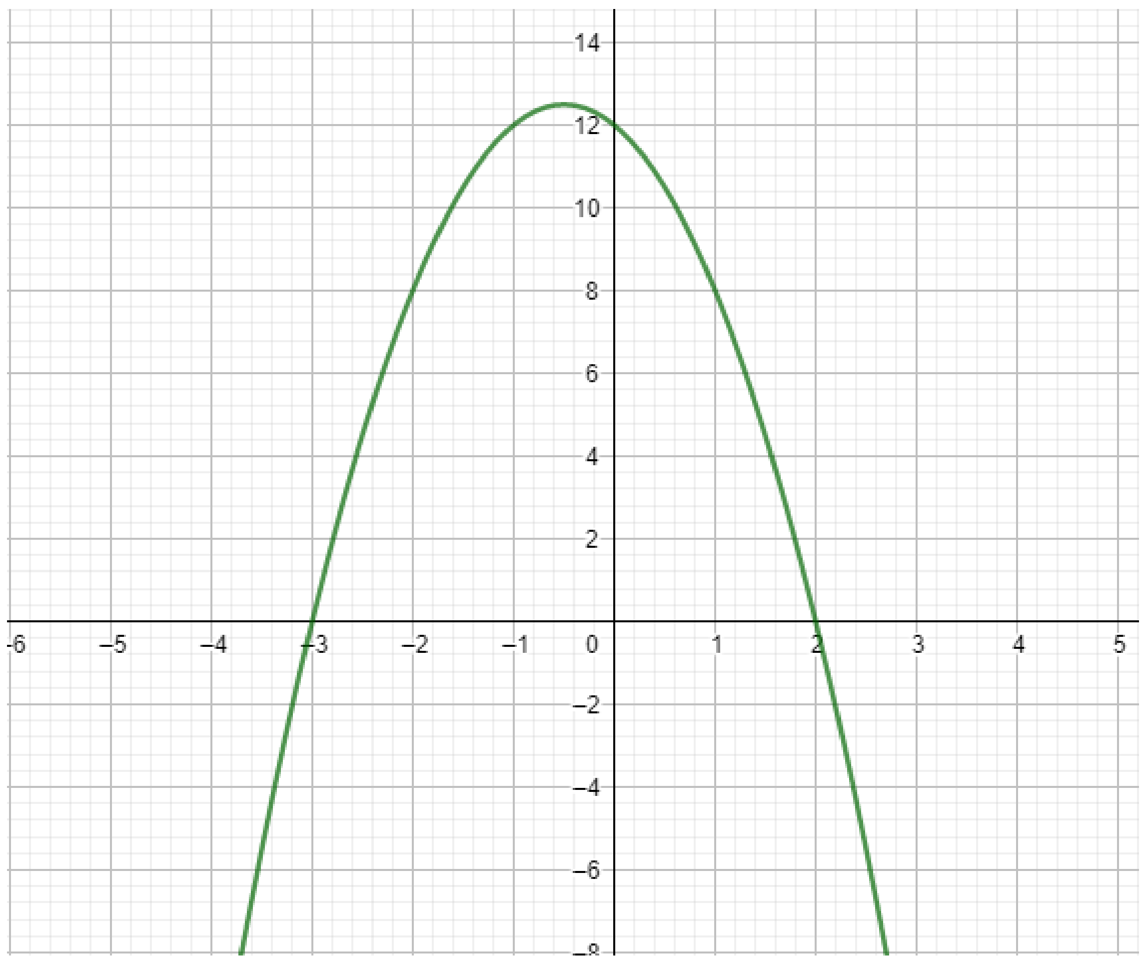
The equation of the graph below can be given in the form $y = ax^2 + bx + c$. Find the values of a, b and c.



A

The y-intercept is 12, so the value of c must be 12.

The x-intercepts are -3 and 2, so when factorised the equation must be $y = k(x+3)(x-2)$.

If we expand this we get $y = kx^2 + kx - 6k$.

But we know $-6k = 12$ so $k = -2$

So the equation is $y = -2x^2 - 2x + 12$

$$\frac{(3^x \times 9^{-x})^2}{27^{1+2x}} = \frac{1}{3}$$

Find the value of x.

B

Notice that 9 and 27 are powers of three, so rewriting the left hand side in terms of powers of 3 we get

$$\frac{(3^x \times (3^2)^{-x})^2}{(3^3)^{1+2x}}$$

Applying index rules, this simplifies

$$= \frac{(3^x \times 3^{-2x})^2}{3^{3+6x}}$$

$$= \frac{(3^{-x})^2}{3^{3+6x}}$$

$$= 3^{-2x-(3+6x)}$$

$$3^{-8x-3}$$

But $\frac{1}{3} = 3^{-1}$ so $-8x - 3 = -1$

That is, $x = -\frac{1}{4}$

B - Solution

Determine the number of distinct values of x that satisfy the equation:

(A) $(2x+3)^{3x+2}=1$

(B) $(3x+2)^{2x+3}=1$

(C) $(2x+3)^{2x+3}=1$

(D) $\left(\frac{2}{x}+1\right)^{\frac{1}{x}+3}=1$

C

If $a^b = 1$ then either

case 1: $b = 0$ and $a \neq 0$ as 0^0 is undefined.

or **case 2:** $a = 1$.

or **case 3:** $a = -1$ and b is even.

(A) **Case 1:** $3x + 2 = 0$ so $x = -\frac{2}{3}$ and when $x = -\frac{2}{3}$, $2x + 3 = \frac{5}{3} \neq 0$ so $x = -\frac{2}{3}$.
Case 2: $2x + 3 = 1$ so $x = -1$.
Case 3: $2x + 3 = -1$ so $x = -2$ and when $x = -2$, $3x + 2 = -4$ which is even, so $x = -2$.
There are three possible values of x ($x = -2, -1, -\frac{2}{3}$).

(B) **Case 1:** $2x + 3 = 0$ so $x = -\frac{3}{2}$ and when $x = -\frac{3}{2}$, $3x + 2 = -\frac{5}{2} \neq 0$ so $x = -\frac{3}{2}$.
Case 2: $3x + 2 = 1$ so $x = -\frac{1}{3}$.
Case 3: $3x + 2 = -1$ so $x = -1$ but when $x = -1$, $2x + 3 = 1$ which is odd, so $x \neq -1$.
There are two possible values of x ($x = -\frac{3}{2}, -\frac{1}{3}$).

(C) **Case 1:** If $2x + 3 = 0$ then this becomes 0^0 which is not well defined, so $2x + 3 \neq 0$.
Case 2: $2x + 3 = 1$ so $x = -1$.
Case 3: If $2x + 3 = -1$ then this becomes $(-1)^{-1} = -1$ so $2x + 3 \neq -1$
There is one possible value of x ($x = -1$).

(D) **Case 1:** $\frac{1}{x} + 3 = 0$ so $x = -\frac{1}{3}$ and when $x = -\frac{1}{3}$, $\frac{2}{x} + 1 = -5 \neq 0$ so $x = -\frac{1}{3}$.
Case 2: $\frac{2}{x} + 1 = 1$ has no solutions.
Case 3: $\frac{2}{x} + 1 = -1$ so $x = -1$ and when $x = -1$, $\frac{1}{x} + 3 = 2$ which is even, so $x = -1$.
There are two possible values of x ($x = -1, -\frac{1}{3}$).

C - Solution

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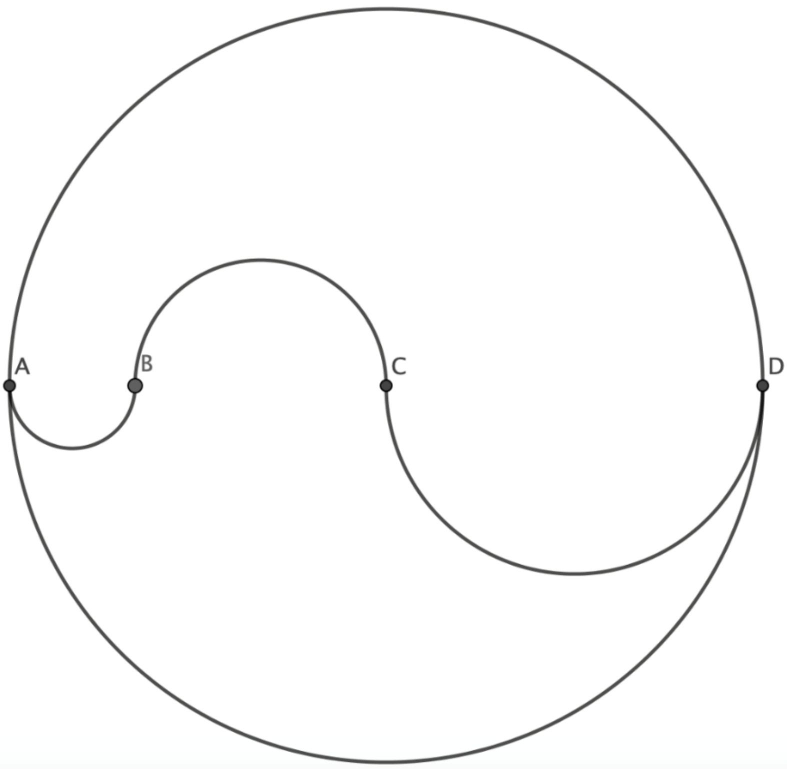
The diagram shows a circle divided into two sections. AD is a diameter of the circle.

You are given that B and C lie on AD such that AB:BC:CD = 1:2:3.

Given that each arc shown is semi-circular, determine the area of the smaller section of the circle as a fraction of the whole circle.



D

Let AB = 2r so the area of the semi-circle with diameter AB is $\frac{\pi r^2}{2}$.

BC = 4r so the area of the semi-circle with diameter BC is $\frac{\pi(2r)^2}{2} = 2\pi r^2$.

CD = 6r so the area of the semi-circle with diameter CD is $\frac{\pi(3r)^2}{2} = \frac{9\pi r^2}{2}$.

AD = 12r so the area of the semi-circle with diameter BC is $\frac{\pi(6r)^2}{2} = 18\pi r^2$.

The total area of the smaller section is therefore

$$18\pi r^2 + 2\pi r^2 - \frac{9\pi}{2} r^2 - \frac{\pi}{2} r^2 = 15\pi r^2.$$

The area of the circle is $\pi(6r)^2 = 36\pi r^2$.

So the fraction is $\frac{15\pi r^2}{36\pi r^2} = \frac{5}{12}$.

D - Solution

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I have a bag with 7 red counters and 3 blue counters. The rules of the game state that if I take a blue counter I have to return it to the bag, but if I take a red counter, I get to keep it.

I play the game three times.

What is the probability I take two red counters and one blue counter?

E

I could take either RRB, RBR or BRR in those orders.

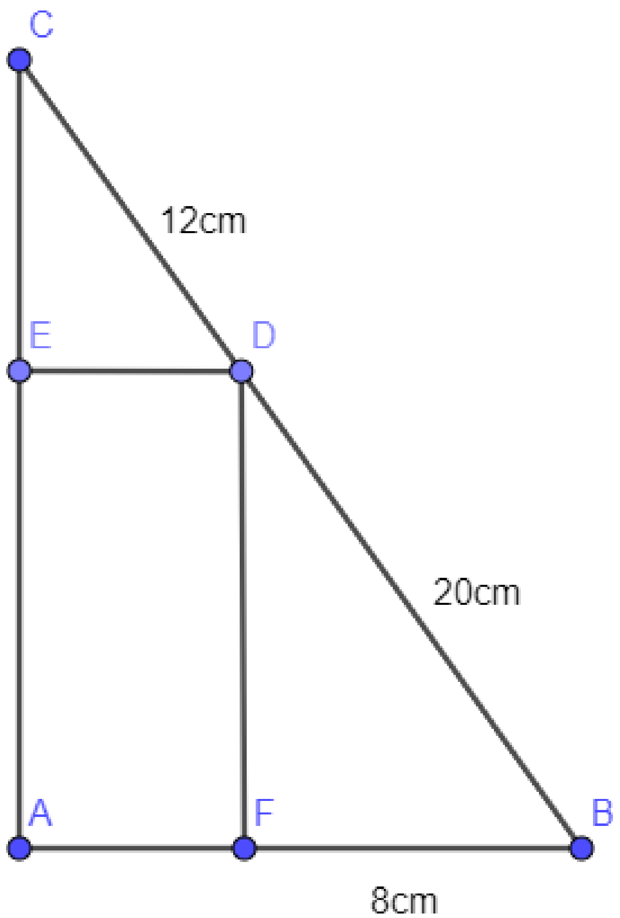
The probability of RRB is $\frac{7}{10} \times \frac{6}{9} \times \frac{3}{8} = \frac{7}{40}$.

The probability of RBR is $\frac{7}{10} \times \frac{3}{9} \times \frac{6}{9} = \frac{7}{45}$.

The probability of BRR is $\frac{3}{10} \times \frac{7}{10} \times \frac{6}{9} = \frac{7}{50}$.

Therefore the total probability is $\frac{7}{40} + \frac{7}{45} + \frac{7}{50} = \frac{847}{1800}$

The diagram shows a rectangle enclosed in a right-angled triangle. CD = 12cm, DB = 20cm and FB = 8cm. Find the area of the rectangle.



F

By Pythagoras' Theorem, $DF = \sqrt{336} = 4\sqrt{21}$.

As triangles CED and DFB are similar (how do you know?) $ED = 8 \times \frac{12}{20} = \frac{24}{5}$.

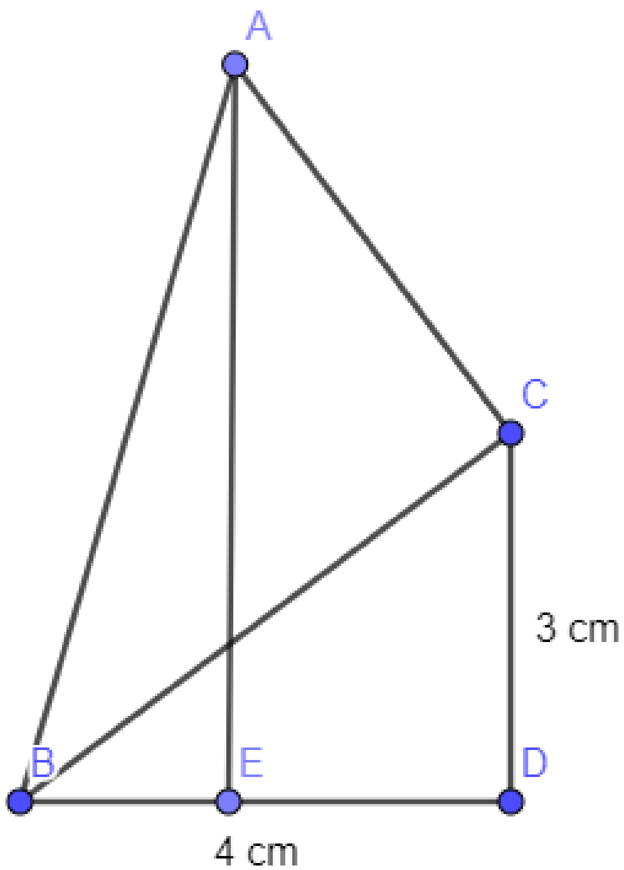
So the area is $\frac{24}{5} \times 4\sqrt{21} = \frac{96\sqrt{21}}{5}$ cm²

The diagram shows two right angled-triangles ABC and CBD, where $\angle ACB$ and $\angle CDB$ are right angles.

Also $\angle ABC = \angle CBD$, $CD = 3$ cm, $BD = 4$ cm.

AE is perpendicular to BE.

Find the lengths of AB and AE.



G

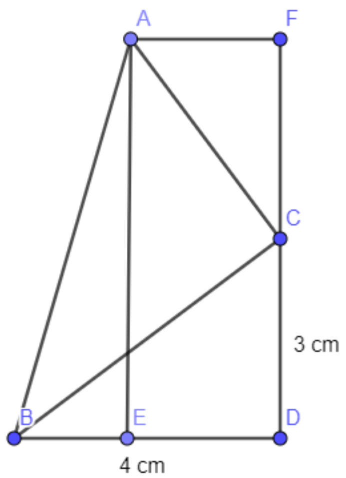
By Pythagoras' Theorem, BC = 5cm.

As ABC and CBD are similar with scale factor $\frac{5}{4}$, $AC = 3 \times \frac{5}{4} = \frac{15}{4}$ and $AB = 5 \times \frac{5}{4} = \frac{25}{4}$

If we draw in another triangle ACF (see below) and let $\angle ABC = \angle CBD = x$, we note $\angle BCD = 90 - x$ and $\angle ACB = 90$ and so $\angle ACF = 180 - 90 - (90 - x) = x$.

Therefore triangle ACF is similar to ABC and CBD, and as we know $AC = \frac{15}{4}$ and $BC = 5$ the scale factor between CBD and ACF is $\frac{\frac{15}{4}}{5} = \frac{3}{4}$ and therefore $CF = 4 \times \frac{3}{4} = 3$.

So $AE = CD + CF = 3 + 3 = 6$ cm



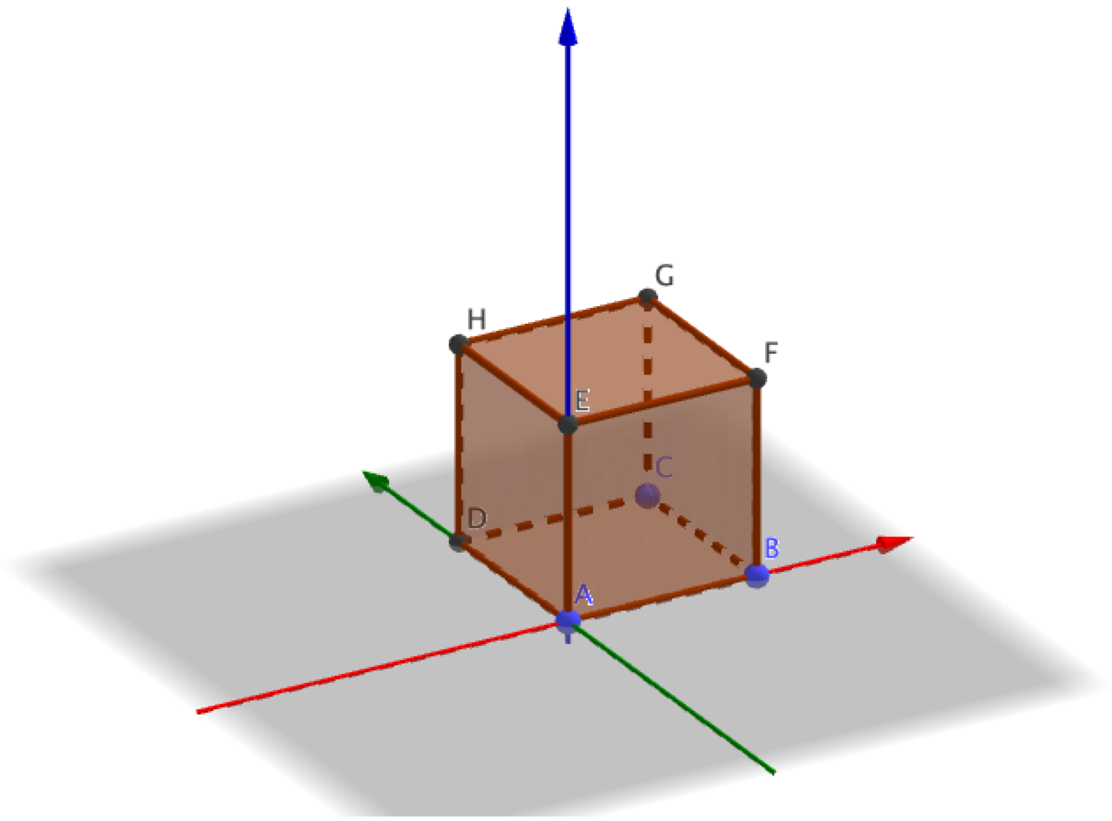
G - Solution

The cube shown has vertices A, B, C, D, E, F, G, and H. To start with the coordinates of the vertices are (0,0,0), (1,0,0), (1,1,0), (0,1,0), (0,0,1), (1,0,1), (1,1,1) and (0,1,1) respectively.

a) The cube is reflected in the face ADHE and then rotated 90° clockwise, looking down the vertical axis. What are the coordinates of the vertices now? Does it matter what order this series of transformations is performed in?

b) The cube is rotated 180° around the edge AB, then 180° around the edge AE, then then 180° degree around the edge AD. What are the coordinates of the vertices now? Does it matter what order this series of rotations is performed in?

c) The cube is reflected in the face ADHE, then the face ABCD then the face ABFE. What are the coordinates of the vertices now? What single transformation is this equivalent to? Does it matter what order this series of rotations is performed in?



H

a) The cube is in its original position, but has been reflected in the plane ACGE (ie. H and F switch and D and B swich). If the order is reversed, the result is the same.

b) All vertices of the cube are in their original locations. This is true for the three rotations performed in any order.

c) The vertices are (0,0,0), (-1,0,0), (-1,-1,0), (0,-1,0), (0,0,-1), (-1,0,-1), (-1,-1,-1) and (0,-1,-1). This holds for any order of transformations. This is equivalent to an enlargement scale factor -1 through the origin (or reflection in the plane $x + y + z = 0$).

In a game of Yip there are 9 players on a team, of which 4 must be Yaps, 3 must be Yups and 2 must be Yops. In the squad there are 6 Yaps, 6 Yups and 5 Yops. How many different teams can I pick from the squad?

In a game of Yep, there are 4 players on a team, all of which are either Yaps, Yups or Yops. A team must include either exactly three Yups, three Yops or three Yaps. In the squad there are 6 Yaps, 6 Yups and 5 Yops. How many different teams can I pick for the squad?

I

In a game of Yip there are

$$\frac{6!}{4!\,2!} \times \frac{6!}{3!\,3!} \times \frac{5!}{3!\,2!} = 15 \times 20 \times 10 = 3000$$

possible teams.

In a game of Yup there are

$$\frac{6!}{3!\,3!} \times 6 \times 2 + \frac{6!}{3!\,3!} \times 5 \times 2 + \frac{5!}{3!\,2!} \times 6 \times 2$$
$$= 20 \times 6 \times 2 + 20 \times 5 \times 2 + 10 \times 6 \times 2 = 560$$

possible teams.

In each of the following, N is an integer. Find the largest possible values of P, Q, R, S and T, and the value of N for which this occurs.

$$P = 18N - N^2 - 76$$

$$Q = 18N^2 - N^4 - 76$$

$$R = -18N^2 - N^4 - 76$$

$$S = 18N - 4N^2 - 76$$

$$T = 18N^3 - N^6 - 76$$

J

$P = 5 - (N - 9)^2$ so the largest possible value of P is 5 when N = 9

$Q = 5 - (N^2 - 9)^2$ so the largest possible value of Q is 5 when $N^2 = 9$, so $N = \pm 3$.

N^2 and N^4 are always non-negative, so the greatest value of R occurs when N = 0, and so the largest possible value of R is - 76. This can also be seem by completing the square (but it is not necessary): $R = 5 - (N^2 + 9)^2$ so as N^2 is always non-negative, $(N^2 + 9)^2$ is minimised when N=0, so R = - 76

$S = -4\left(N - \frac{9}{4}\right)^2 - \frac{223}{4}$ so S is greatest when N = 9/4. However, N must be an integer. As this is quadratic, the nearest integer to 9/4 will give the greatest value of S. This is N =2, and so S = -56

$T = 5 - (N^3 - 9)^2$ so the largest possible value of T occurs when $N^3 = 9$ and so $N = \sqrt[3]{9}$. However, N is an integer, so N is the nearest integer to $\sqrt[3]{9}$ which is 2. When N = 2, T = 4. We can verify this by noting when N=3, T=- 319.

J – Solution

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When $x = 6$, $x^2 + 2x - 32 = 16$, which is a square number.

Are there any other integer values of x for which $x^2 + 2x - 32$ is a square number?

K

$$x^2 + 2x - 32 = y^2$$
$$(x + 1)^2 - 33 = y^2$$
$$(x + 1)^2 - y^2 = 33$$
$$(x + 1 - y)(x + 1 + y) = 33$$

So either $x + 1 + y = 33$ and $x + 1 - y = 1$ (or vice versa, which gives the same solution)

Or $x + 1 + y = 11$ and $x + 1 - y = 3$ (or vice versa, which gives the same solution)

Solving simultaneously in the first case gives x = 16, and in the second case x = 6. So x = 16 is the only other value of x.

K – Solution

Calculate $(\pi^3 - \pi)\left(\frac{1}{\pi^2-\pi} - \frac{1}{\pi^2+\pi}\right)$.

L

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$$(\pi^3 - \pi)\left(\frac{1}{\pi^2 - \pi} - \frac{1}{\pi^2 + \pi}\right)$$

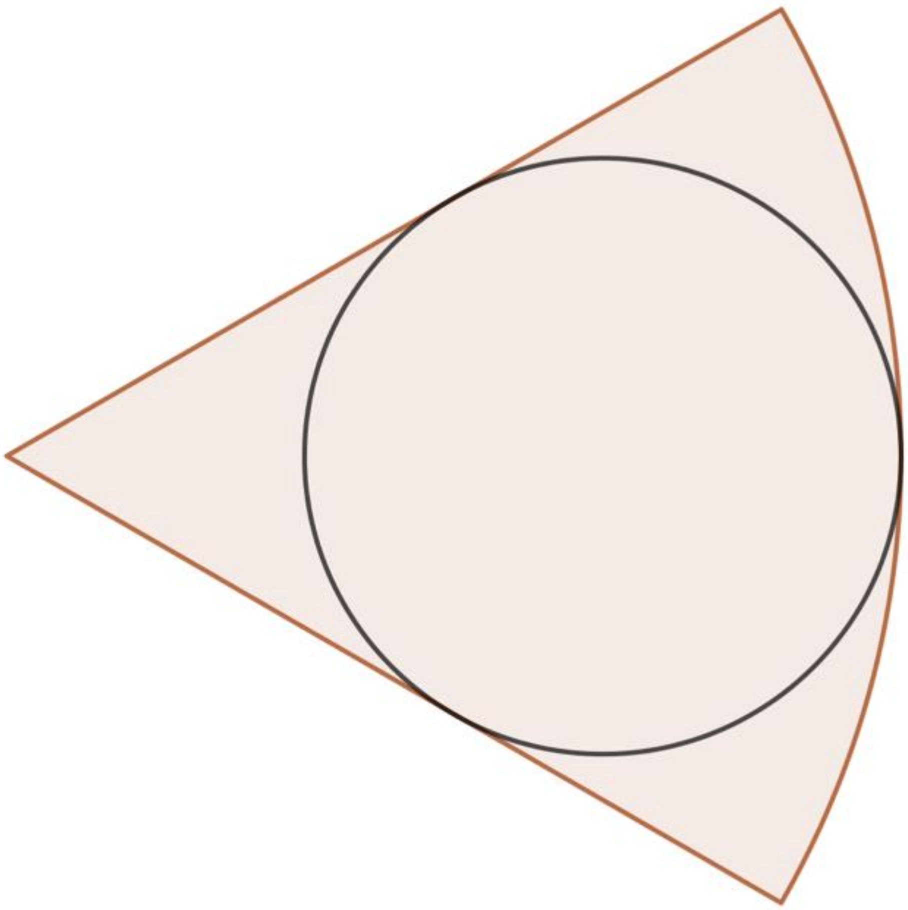
$$= \pi(\pi + 1)(\pi - 1)\left(\frac{1}{\pi(\pi - 1)} - \frac{1}{\pi(\pi + 1)}\right)$$

$$= \left(\frac{\pi(\pi + 1)(\pi - 1)}{\pi(\pi - 1)} - \frac{\pi(\pi + 1)(\pi - 1)}{\pi(\pi + 1)}\right)$$

$$= \pi + 1 - (\pi - 1) = 2$$

L - Solution

The sector shown is one sixth of a circle. A second circle is inscribed in the sector (so that it touches both the arc and the two straight sides). What fraction of the sector is covered by the interior of the circle?



M

Let the radius of the sector be x.

Let the radius of the circle be r so that the distance from the centre of the larger circle to the centre of the smaller circle is $x - r$. Then the centres of the two circles and the point where the circle touches the straight side forms a right-angled triangle so that $\sin 30 = \frac{r}{x-r}$.

That is $\frac{1}{2} = \frac{r}{x-r}$

$x - r = 2r$

$x = 3r$

So the area of the circle is πr^2 and the area of the sector is $\frac{1}{6}\pi(3r)^2 = \frac{3}{2}\pi r^2$ and so the fraction occupied by the circle is $\frac{\pi r^2}{\frac{3}{2}\pi r^2} = \frac{2}{3}$